Industrial Automation Project Ts: Shannon Nyquist Theorem

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1 Introduction

The Shannon Nyquist Theorem bridges a link between continuous time signals and discrete time signals. It gives the relation between the reconstruction of the signal and the frequency of sampling of the original signal. These results are most useful while sampling music for storing. We humans can hear sounds of maximum frequency from 20Hz and 20,000Hz hence music is sampled at least 40,000Hz to recreate it without any losses. This report will explain this phenomenon with some examples.

2 Sampling and Reconstruction

In Physical Systems an Analog to Digital converter send an impulse at the specified frequency.

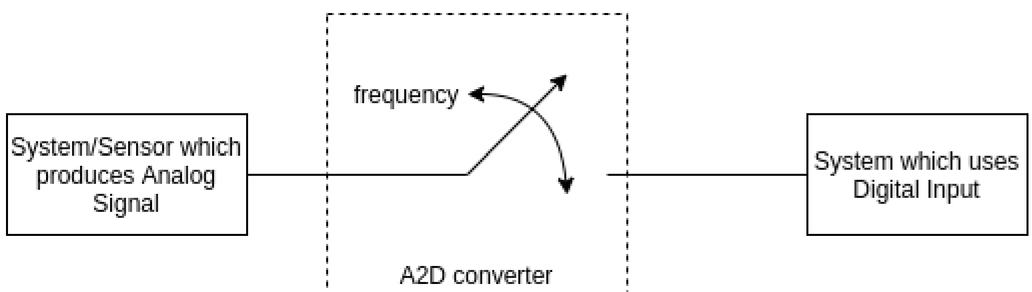

Figure 1: Analog to Digital converter Block Diagram

Sampling time = 0.17 and 0.017 secs which implies that the sampling frequency = 5.882 and 58.824 Hz respectively.

In the code I took the value of the original function which is:

$$x = \sin 2\pi t + 0.2 \cos 12\pi t$$

at every time period which led to a discrete signal.[1]

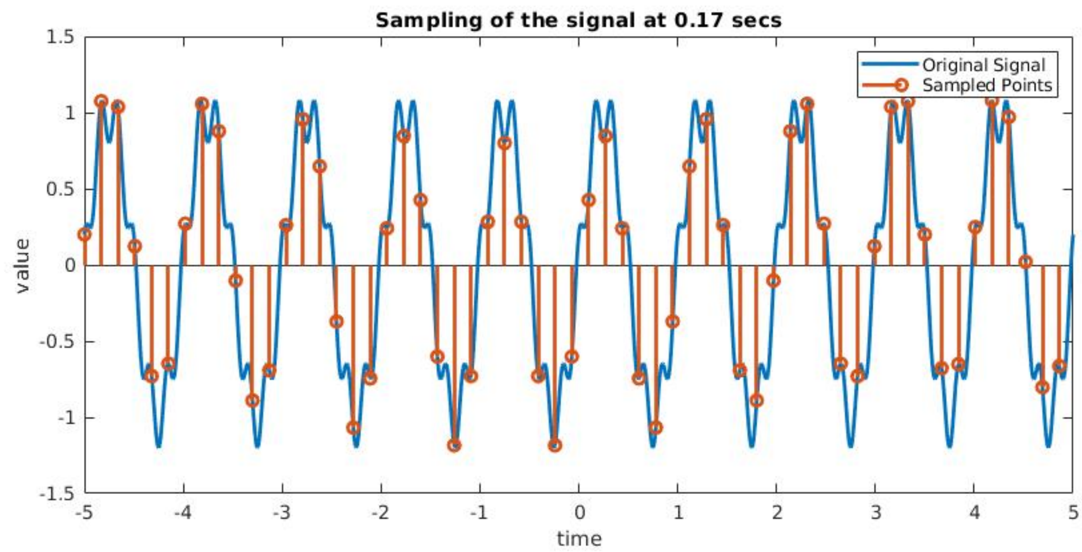

Figure 2: Sampling rate = 0.17 secs

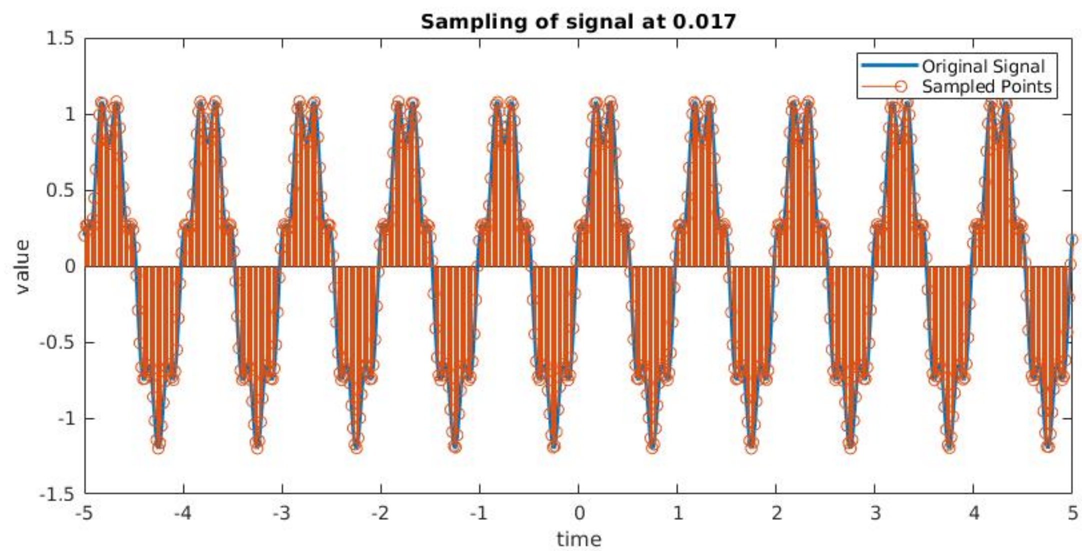

Figure 3: Sampling rate = 0.017 secs

For reconstructing the signal we convolute the impulse response of the filters with the discrete signal.

Impulse is a signal with a sharp rise at $t = 0$ and falls again at $t = 0$.

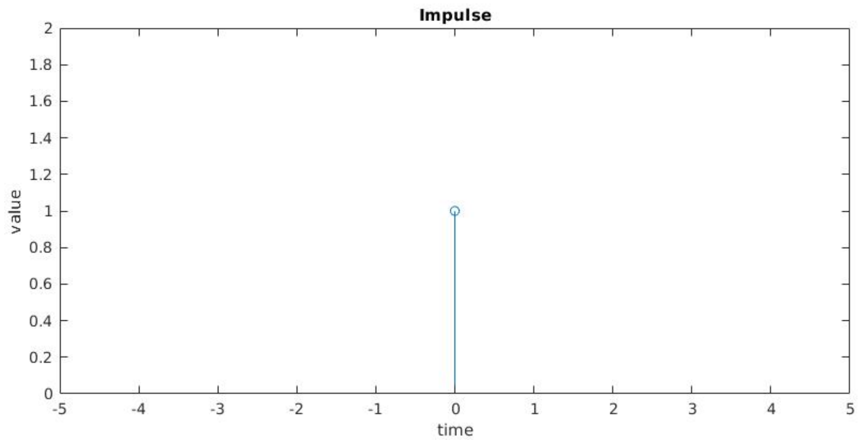

Figure 4: Impulse Signal

The reconstruction will be done using 4 techniques namely:

1. Ideal Low Pass filter

   Low pass filter can be best represented in the frequency domain as:

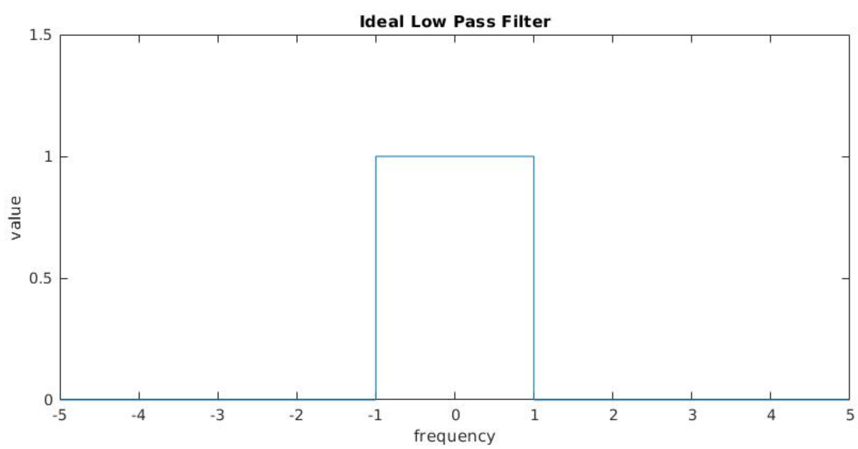

Figure 5: Ideal Low Pass Filter in Frequency Domain

If we take an inverse fourier transform of the above signal we get the impulse response of the ideal low pass filter in time domain.

$$Impulse\ Response = \int_{-1}^{1} e^{i2ft} df$$

$$Impulse\ Response = \frac{2\sin 2\pi t}{2\pi t}$$

$$Impulse\ Response = 2\,\mathrm{sinc}\, 2\pi t$$

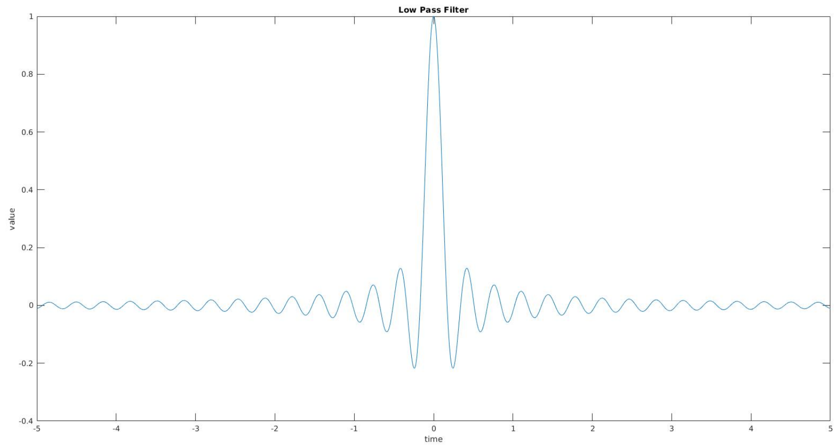

Figure 6: Ideal Low Pass Filter Impulse Response

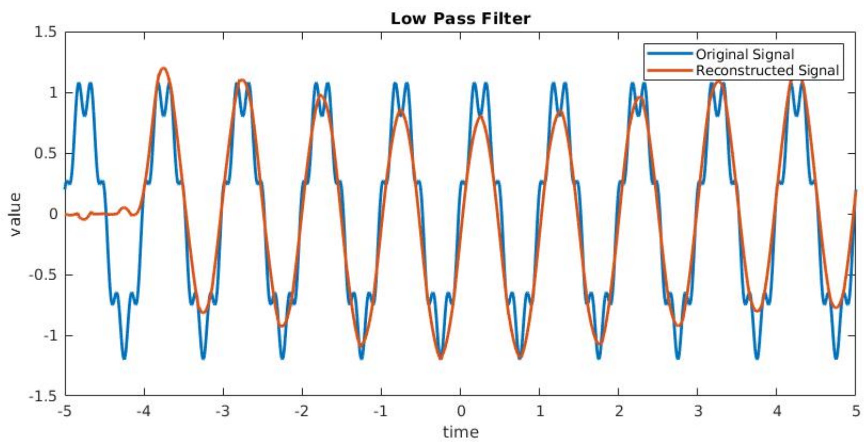

Figure 7: Reconstruction using Ts = 0.17

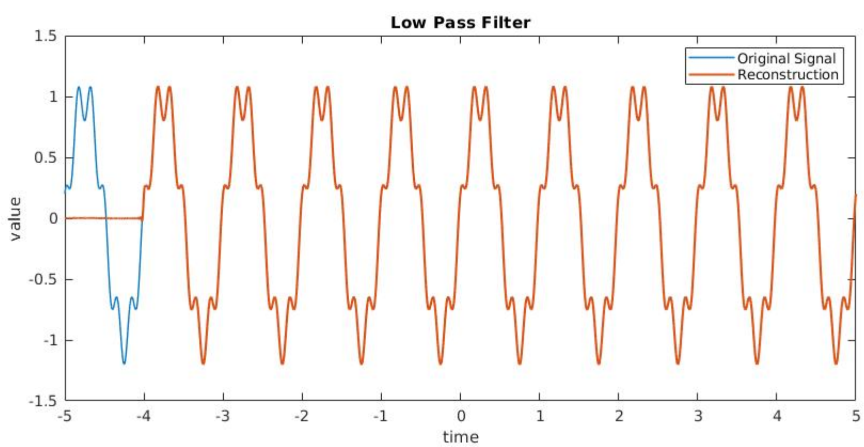

Figure 8: Reconstruction using Ts = 0.017

2. Zero Order Hold

   In zero order hold the value of the signal at a time step (k) is held till the next time step and the signal drops or rises to the next value of time step $(k+1)$.

   Let $\hat{x}(t)$ be the reconstructed signal and $X(k)$ be the discrete values at time steps.

$$\hat{x}(t) = X(k) \qquad [k * Ts < t < (k+1) * Ts]$$

$$\{X(k) = 1 \qquad [0 < k < 1]\}$$

$$\hat{x}(t) = 1 \qquad [0 < t < Ts]$$
$$= 0 \qquad otherwise$$

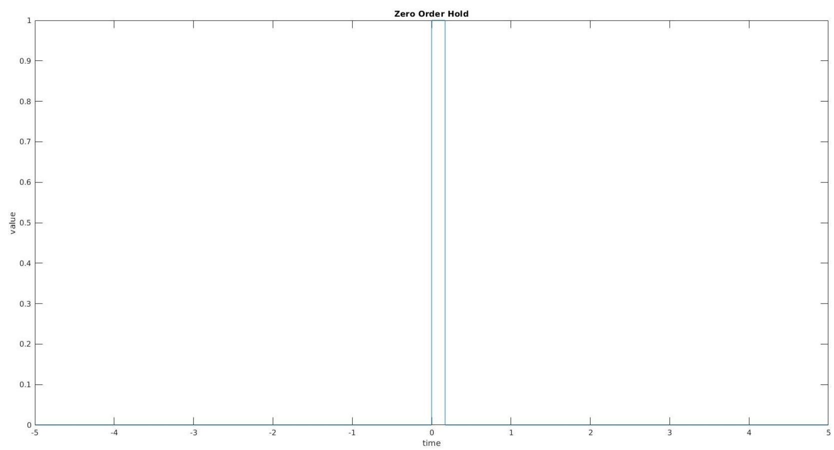

Figure 9: Zero Order Hold Impulse Response

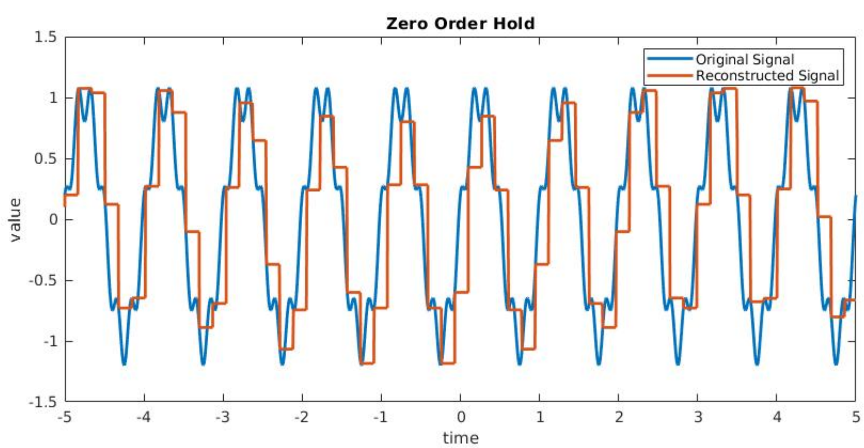

Figure 10: Reconstruction using Ts = 0.17

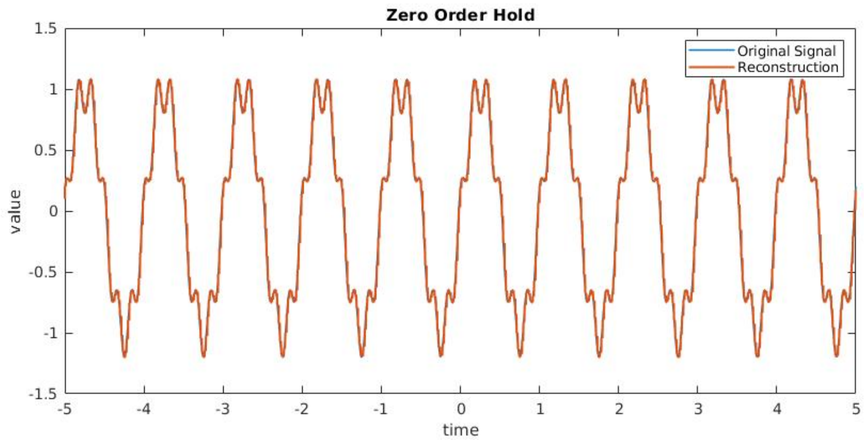

Figure 11: Reconstruction using Ts = 0.017

3. Non Causal First order hold

   In the first order hold the signal is reconstructed by joining the discrete points on the signal at time steps $(k-1)$ and $(k)$by a straight line.

   Let $\hat{x}(t)$ be the reconstructed signal and $X(k)$ be the discrete values at time steps.

$$\hat{x}(t) = \frac{X(k+1) - X(k)}{Ts}(t - k * Ts) + X(k) \quad [k * Ts < t < (k+1) * Ts]$$

$$\{X(k+1) = 1,\ X(k) = 0 \quad [-1 < k < 0]$$

$$X(k+1) = 0,\ X(k) = 1 \quad [0 < k < 1]\}$$

$$\hat{x}(t) = \frac{1-0}{Ts}(t + Ts) + 0$$

$$\hat{x}(t) = \frac{t}{Ts} + 1 \quad [-Ts < t < 0]$$

$$\hat{x}(t) = \frac{0-1}{Ts}(t) + 1$$

$$\hat{x}(t) = \frac{-t}{Ts} + 1 \quad [0 < t < Ts]$$

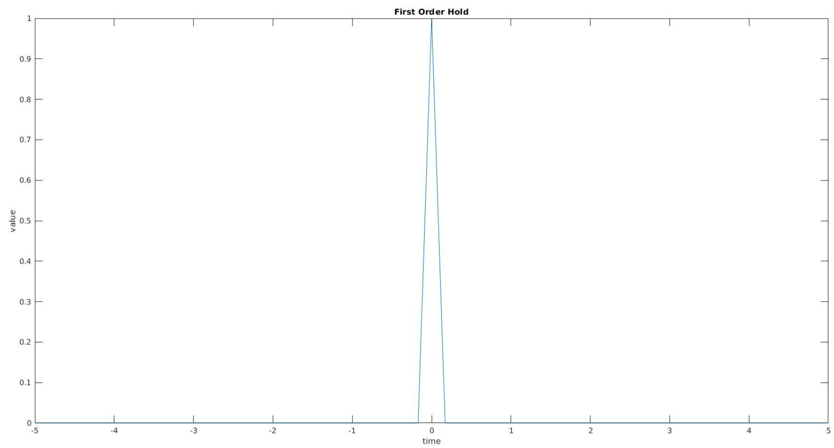

Figure 12: First Order Hold Impulse Response

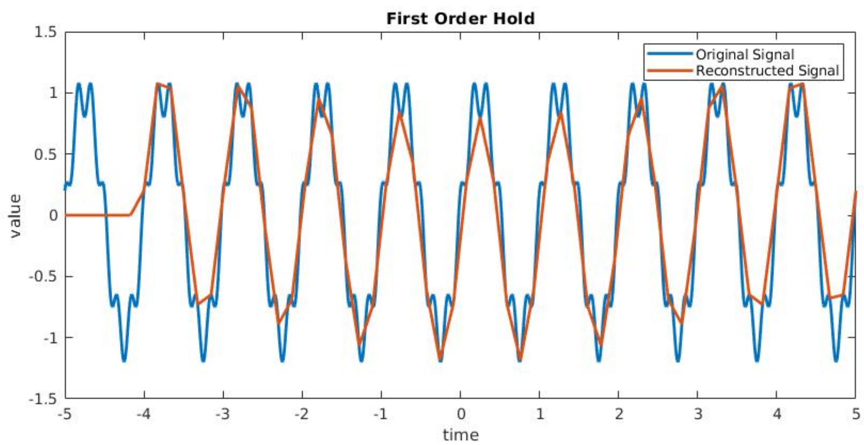

Figure 13: Reconstruction using Ts = 0.17

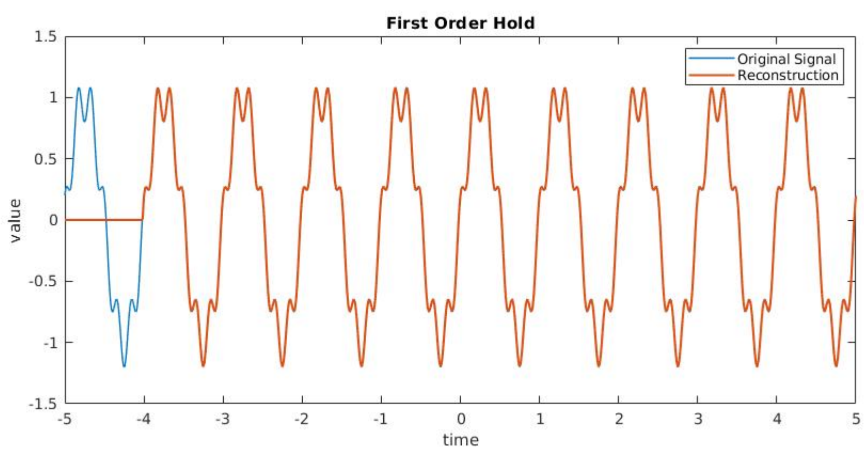

Figure 14: Reconstruction using Ts = 0.017

4. Predictive first order hold

In the predictive first order hold the signal is reconstructed using the slope of the previous two sampled points and extend it till the next time step. This is a causal system.

Let $\hat{x}(t)$ be the reconstructed signal and $X(k)$ be the discrete values at time steps.

$$\hat{x}(t) = \frac{X(k) - X(k-1)}{Ts}(t - k*Ts) + X(k) \quad [k*Ts < t < (k+1)*Ts]$$

$$\{X(k) = 1,\ X(k-1) = 0 \quad [0 < k < 1]$$

$$X(k) = 0,\ X(k-1) = 1 \quad [1 < k < 2]\}$$

$$\hat{x}(t) = \frac{1-0}{Ts}(t+0) + 1$$

$$\hat{x}(t) = \frac{t}{Ts} + 1 \quad [0 < t < Ts]$$

$$\hat{x}(t) = \frac{0-1}{Ts}(t - Ts) + 0$$

$$\hat{x}(t) = \frac{-t}{Ts} + 1 \quad [Ts < t < 2Ts]$$

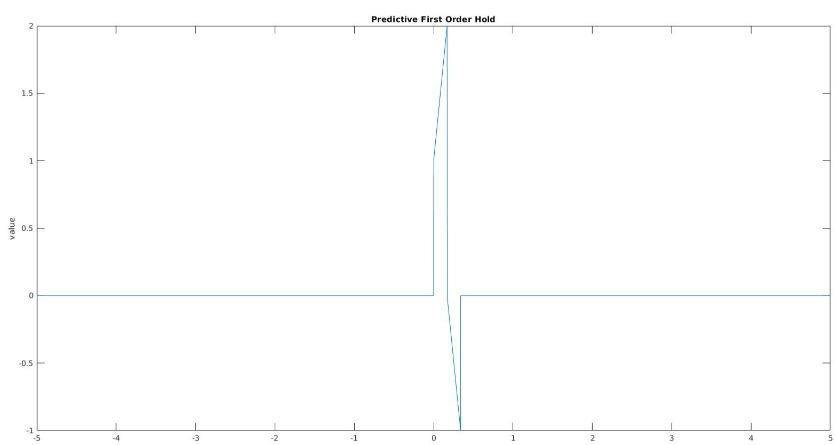

Figure 15: Predictive First Order Hold Impulse Response

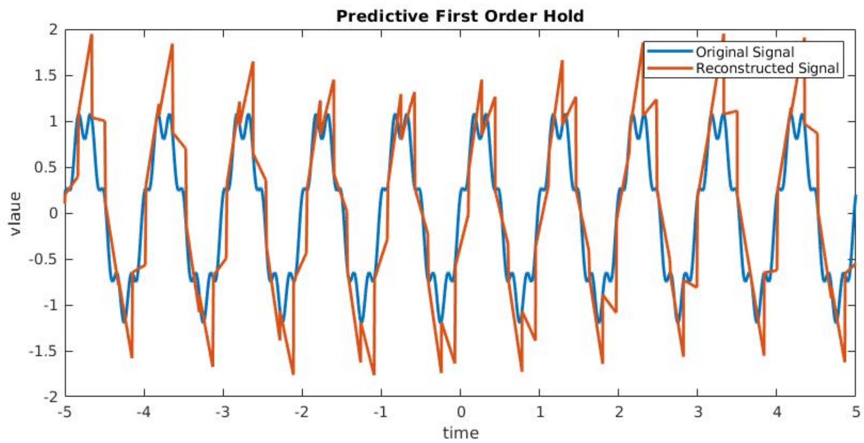

Figure 16: Reconstruction using Ts = 0.17

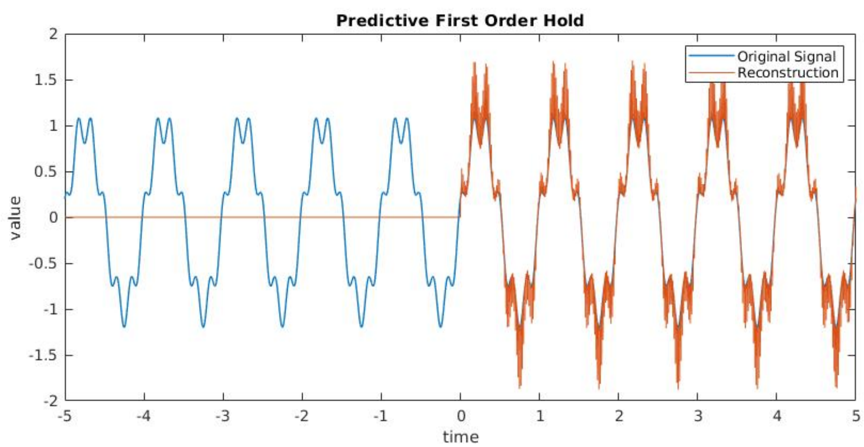

Figure 17: Reconstruction using Ts = 0.017

Comparing the Fourier Response Function of Impulse Response of Filters.
Derivation of Fourier Transfer of Impulse responses

1. Ideal Low Pass Filter

   It is a step function till the cutoff frequency and is zero for the rest of the frequencies.

2. Zero Order Hold

$$\mathscr{F}(ZeroOrderHold) = \int_{-\frac{Ts}{2}}^{\frac{Ts}{2}} 1 * e^{-2\pi ift} dt$$

$$= \frac{e^{-2\pi ift}}{-2\pi if}\Big|_{-Ts/2}^{Ts/2}$$

$$= \frac{1}{-2\pi if}[e^{-\pi ifTs/2} - e^{\pi ifTs/2}]$$

$$= \frac{Ts/2}{\pi f Ts/2}[\sin \pi f Ts/2]$$
$$= \frac{Ts}{2} \operatorname{sinc} \frac{fTs}{2}$$

3. First Order Hold

$$\mathscr{F}(FirstOrderHold) = \int_{-\frac{Ts}{2}}^{0} (1 + 2t/Ts)e^{-2\pi i f t}dt + \int_{0}^{\frac{Ts}{2}} (1 - 2t/Ts)e^{-2\pi i f t}dt$$
$$= [\frac{1 + 2\pi i f Ts/2}{4\pi^2 (fTs/2)^2} - \frac{e^{2\pi i f Ts/2}}{4\pi^2 f^2}] - [\frac{2\pi i f Ts/2 - 1}{4\pi^2 (fTs/2)^2} - \frac{e^{-2\pi i f Ts/2}}{4\pi^2 f^2}]$$
$$= -\frac{e^{-2\pi i f Ts/2}(e^{2\pi i f Ts/2} - 1)^2}{\pi^2 f^2 Ts^2}$$
$$= -\frac{e^{-2\pi i f Ts/2}(e^{\pi i f Ts/2}[e^{\pi i f Ts/2} - e^{-\pi i f Ts/2}])^2}{\pi^2 f^2 Ts^2}$$
$$= (\frac{\sin(\pi f Ts/2)}{\pi f Ts/2})^2$$
$$= (\operatorname{sinc} \frac{fTs}{2})^2$$

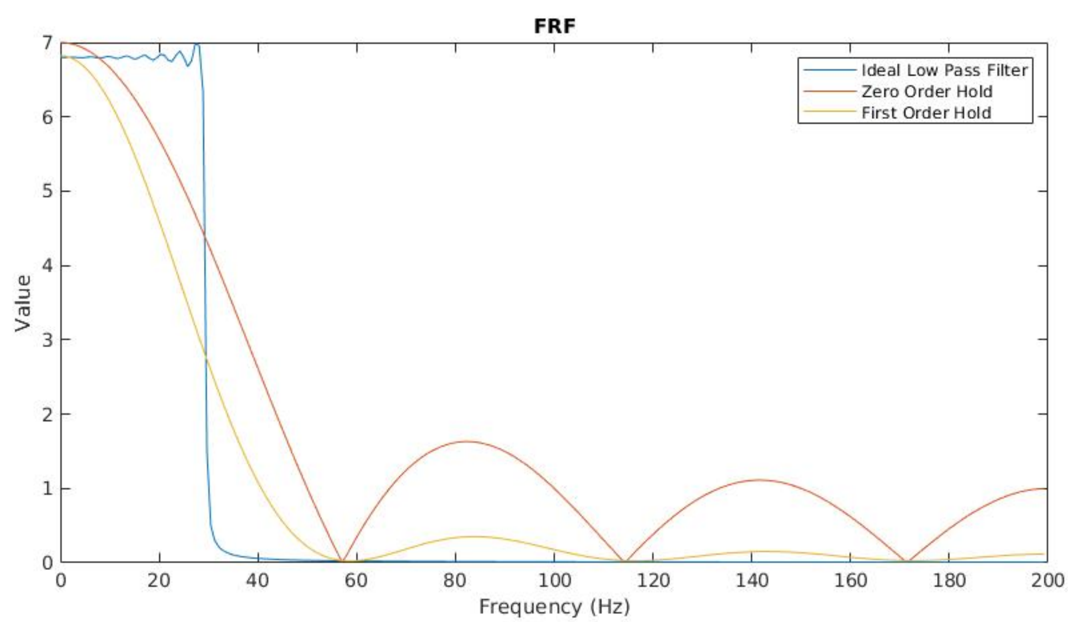

Figure 18: Reconstruction using Ts = 0.017

Comparing from the above figure the higher frequencies from the first order hold are cutoff as the amplitudes of the subsequent lobes are less than zero order hold.

The ideal low pass filter is as the name suggests cuts off all the higher frequencies. So it is better than the first order hold.

3 Aliasing

Aliasing is a phenomenon where the samples of an original signal can be the samples of other signals with low or high frequencies. The high frequencies should not be considered as there are countless harmonics.[2]

We can observe when the sampling rate is low there is a low frequency wave added with the reconstructed signal. This phenomenon is observed clearly in the ideal low pass filter where a high frequency $f = 12$ is cut off from the reconstructed signal.

The two sampling rate we used are 5.882 and 58.824 Hz. The first frequency is less than the highest frequency in the original signal. Hence we will get an aliased reconstructed signal. The second frequency is greater than twice the maximum signal frequency. Hence the signal is reconstructed correctly.

If we consider the reconstructed signal using low pass filter we can see the frequency of 1 Hz and also another low frequency wave. This frequency can be calculated by using the frequency folding along the nyquist frequency.

Folding of the frequency at the sampling and nyquist frequency is represented as:

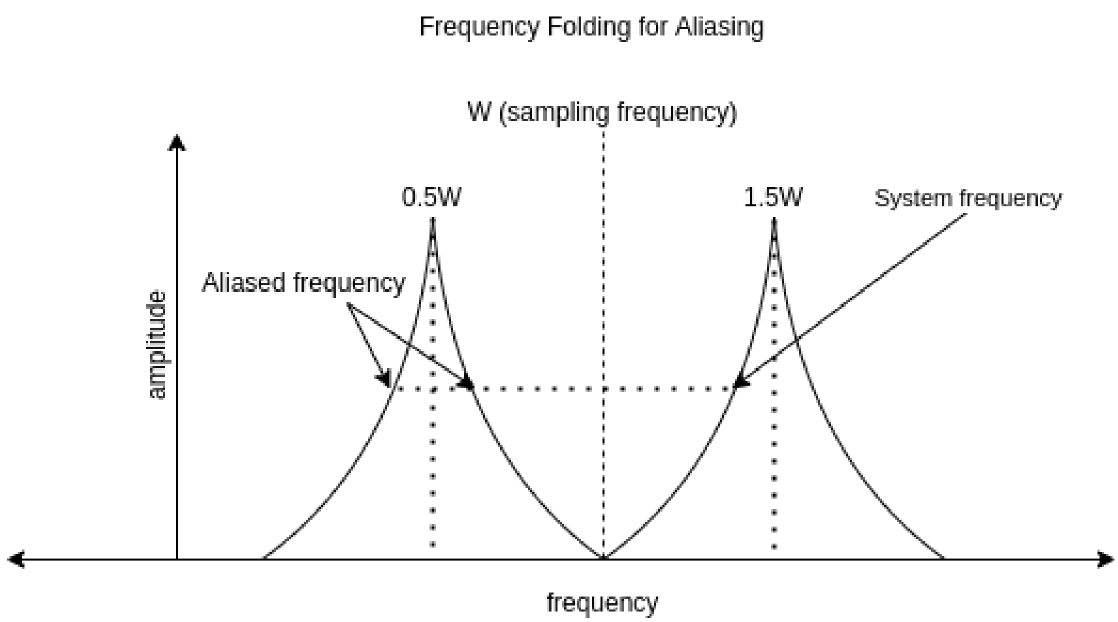

Figure 19: Folding frequency

4 Shannon-Nyquist Theorem and its Visualisation

"A function containing no frequency higher than ωHz, is completely determined by sampling at 2ωHz."[3]

Its corollary can be implied as:

"To resolve all frequencies is an function, it must be sampled at twice the highest frequency present."

This frequency 2ωHz is now know as the Nyquist rate.

This result is one of the most important results in the information theory. Sampling of a signal is used to pass on the information with the least possible memory and reconstruction is the technique used to retrieve that information without any loss.

From the above examples we used only two sampling rates to reconstruct the signal. But the visualisation of Shannon Nyquist Theorem is best represented with a lot of different sampling rates.

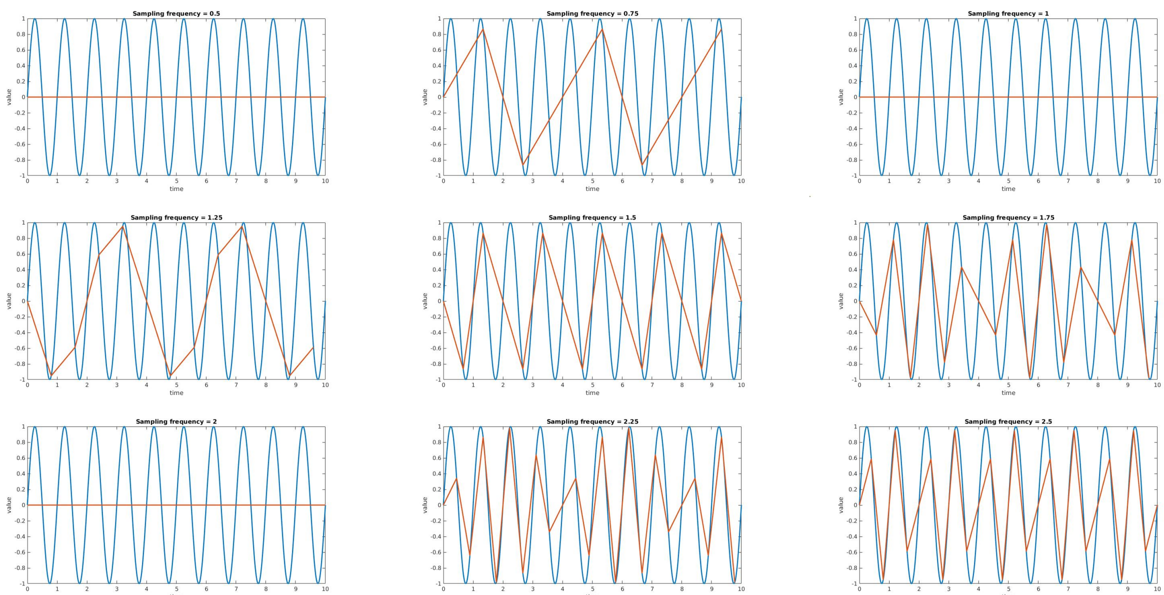

Figure 20: Increasing sampling rate gradually

From this example we can conclude the theorem as the frequency of the reconstructed signal becomes equal to the original when the signal is sampled at ore than twice the original frequency. But the signal is not reconstructed as correctly as the original one. For that we can increase the sampling rate further as twice rate is only a theoretical value for the perfect reconstruction.

References

[1] Harry Nyquist. "Certain topics in telegraph transmission theory". In: *Transactions of the American Institute of Electrical Engineers* 47.2 (1928), pp. 617–644.

[2] *Shannon Nyquist Sampling Theorem - YouTube*. https://www.youtube.com/watch?v=FcXZ28BX-xE&ab_channel=SteveBrunton. (Accessed on 03/12/2021).

[3] Claude E Shannon. "A mathematical theory of communication". In: *The Bell system technical journal* 27.3 (1948), pp. 379–423.

Appendices

Matlab code for Sampling and Recostruction

```
clear
clc
close all

Ts=0.17; %samples increment

dt=0.0001; %continous increment

lowerLimit=-5;
upperLimit=5;

t=lowerLimit:dt:upperLimit; %limits

x = @(t)(sin(2*pi*t)+0.2*cos(12*pi*t));

% sampling

tSample=lowerLimit:Ts:upperLimit;

xSample=@(tSample)(sin(2*pi*tSample)+0.2*cos(12*pi*tSample));

% plot(t,x(t),'LineWidth',2)
% hold on
% stem(tSample,x(tSample),'LineWidth',0.5)

f1=figure();
% f2=figure();
% f3=figure();
f4=figure();

for choice=(1)

if choice==1 % low pass filter

    samples  = 0:length(tSample)-1;

    xs = zeros(1,length(t));
    xs(1:Ts/dt:end) = x(samples*Ts);

%     lpfImpl = @(t) sinc((t)/Ts);
```

```
%     figure(f1)
%
%     subplot(4,1,1)
%
%     plot(t,lpfImpl(t))

    lowPassFilter = @(t) sinc((t-lowerLimit-1)/Ts);

    lowPass = conv(lowPassFilter(t),xs);

    lowPass = lowPass(1:length(t));

    figure(f1)

%     subplot(4,1,1)

    plot(t,x(t),'LineWidth',1)

    hold on

    plot(t,lowPass,'LineWidth',1.5)

end

if choice==2 % zero order

    samples  = 0:length(tSample)-1;

    xs = zeros(1,length(t));
    xs(1:Ts/dt:end) = x(samples*Ts);

    zeroOrderFilter     = @(t) rectangularPulse(0,Ts,t);

    figure(f1)
%
%     subplot(4,1,2)
%
    plot(t,zeroOrderFilter(t))

    zeroOrderFilter     = @(t) rectangularPulse(0,Ts,t-lowerLimit);

    zeroOrder = conv(zeroOrderFilter(t),xs);

    zeroOrder = zeroOrder(1:length(t));
```

```
    figure(f2)

%     subplot(4,1,2)

    plot(t,x(t),'LineWidth',1)

    hold on

    plot(t,zeroOrder,'LineWidth',1.5)

end

if choice==3 % first order

    samples  = 0:length(tSample)-1;

    xs = zeros(1,length(t));
    xs(1:Ts/dt:end) = x(samples*Ts);

    firstOrderFilter     = @(t) triangularPulse((t)/Ts);

    figure(f1)
%
%     subplot(4,1,3)
%
    plot(t,firstOrderFilter(t))

    firstOrderFilter     = @(t) triangularPulse((t-lowerLimit-1)/Ts);

    firstOrder = conv(firstOrderFilter(t),xs);

    firstOrder = firstOrder(1:length(t));

    figure(f3)

%     subplot(4,1,3)

    plot(t,x(t),'LineWidth',1)

    hold on

    plot(t,firstOrder,'LineWidth',1.5)

end
```

```
if choice==4 % predictive first order

    samples  = 0:length(tSample)-1;

    xs = zeros(1,length(t));
    xs(1:Ts/dt:end) = x(samples*Ts);

    pFirstOrderFilter   = @(t) (rectangularPulse((t)/Ts - 1/2) - rectangularPulse((t)/Ts

    figure(f1)

%     subplot(4,1,4)

    plot(t,pFirstOrderFilter(t))

    pFirstOrderFilter   = @(t) (rectangularPulse((t-lowerLimit)/Ts - 1/2) - ...
                        rectangularPulse((t-lowerLimit)/Ts - 3/2) + ...
                        triangularPulse((t-lowerLimit)/Ts - 1));

    pFirstOrder = conv(pFirstOrderFilter(t),xs);

    pFirstOrder = pFirstOrder(1:length(t));

    figure(f4)

%     subplot(4,1,4)

    plot(t,x(t),'LineWidth',1)

    hold on

    plot(t,pFirstOrder)

end

end
```